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TECHNICAL REPORT 2

SOME SEARCH PROBLEMS WITH FALSE CONTACTS

James M. Dobbie

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Prepared By

Arthur D. Little, Inc.
Cambridge, Massachusetts 02140

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SOME SEARCH PROBLEMS WITH FALSE CONTACTS*

James M. Dobbie
Arthur D. Little, Inc., Cambridge, Mass.

ABSTRACT

Some search problems are described and discussed, starting with the case of no false contacts, to provide a foundation for the development of a search theory in the presence of false contacts. Important properties of the search plan are defined and illustrated by examples. After a general description of false contact generators and alternative actions that may be used when contacts are made, the problem is formulated for the case in which false contacts are generated by real stationary objects that are investigated when contacted. It is shown that the formulation and solution of the standard optimization functionals are contingent on the number of false targets found, in general, and possibly on the locations and times of contact as well. The optimization functional is difficult to write and more difficult to solve. The formulation is made for the expected-time functional when the number of false targets is limited to finite values. The solution is outlined and illustrated with an example.

If the region being searched is known to contain exactly one target and no false contacts occur, the formulation of the criterion for the probability of detection with a given effort, or for the expected effort (or time) to find the target, is an easy problem. Also, the solution is not difficult. If the region contains real objects, other than the target, that can generate false contacts, or if false contacts occur from random

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fluctuations, the optimization criterion may be difficult to formulate and very difficult to solve. We demonstrate this fact by formulating the criterion for a class of false contacts, and solving the optimization for a particular example.

The first serious attempt to develop a theory of search in the presence of false targets is that of Stone and Stanshine [4]. They assume that false contacts are generated by real objects that can be marked, when located and identified, so that they will not require investigation should they be contacted again. They use a special set of assumptions that permits the expected-time criterion to be formulated without including the possibility that the optimal search plan may depend on the number of false targets that have been found and eliminated. We show that the optimal search plan does not have this property, in general; and that the expected-time functional, or the probability functional, is difficult to construct, and more difficult to solve. The formulation and solution have been obtained for some particular cases, and a method for the formulation and solution has been outlined for other cases. The general problem remains unsolved.

Some special properties of the optimal solution are discussed in section 1, starting with the case of no false contacts. The general false contact search problem is discussed in section 2 and the problem in which false contacts are generated by real stationary objects is discussed in section 3. Conditional detection functions and posterior false target distributions are developed in sections 4 and 5. After a discussion of optimization criteria in section 6, the problem is formulated for the expected-time criterion in section 7 and solved for a simple two-cell problem in section 8.

1. Additivity, Consistency, and Contingencies

We start with a discussion of some properties of the optimal search plan when the region contains one, and only one, target and no false contacts occur. Koopman [3, p. 617] describes a property of the optimal search

density function that depends on the optimization criterion, as well as the conditional detection function. It states that the optimal search density function that maximizes the probability of detecting the target with a given total effort $E_1 + E_2$, for positive E_1 and E_2, is the sum of the optimal density function for E_1 and the conditionally optimal density function for E_2, applied in the order E_1 followed by E_2. The conditionally optimal density function for E_2 is the optimal density function for E_2, given that the target has not been found with the previous search effort E_1 distributed optimally. If the optimal search density function satisfies this condition for all divisions of the total search effort, we say that it is additive, or that it has the additivity property. If the optimal search density function is additive, we can apply the effort piecemeal, optimizing conditionally for each piece, and the total effort will be optimally distributed.

The optimal search density function usually has another property that sometimes is confused with the additivity property. Suppose that the effort E_1 has been applied unsuccessfully according to the plan that is optimal for the effort $E_1 + E_2$ and we pause to replan the remainder E_2 of the search effort. If the optimal search plan for the remainder is the same as it is in the original search plan and this holds for all divisions of the total effort, we say that the optimal search plan is consistent, or that it has the consistency property.

As the search proceeds, various events may occur as results of the search, and otherwise. The optimal search plan may depend on these contingencies or be independent of them. An obvious contingency is the state of the target detection. Since there is exactly one target in the region being searched, we assume that search stops, if the target is found. Search continues only when the target has not been found (and perhaps not then). Thus, the continuation of search is contingent on the target not being found. If the search plan is independent of all contingencies other than the state of target detection, we say that it is contingent-free.

The additivity, consistency, and contingent-free properties are not identical. The additivity property says that we can't improve the search plan by planning the distribution of the entire effort at the start, rather than doing it piecemeal. The consistency property says that, if we plan optimally, no improvement can be made after part of the search has been made, by considering the results of the searching done thus far. The contingent-free property says that the only contingency that we need to consider in planning the search is the state of target detection.

If the contingencies are limited to the state of target detection and the region is known to contain exactly one target and no false targets, the consistency property is a consequence of optimality, provided the optimal search plan is followed exactly and the "pause" is conceptual only. When a conceptual pause is made to replan the search the only available information is that the target has not been found with the effort applied thus far, which is the condition assumed to hold for the continuation of search. If the optimal search plan has been followed exactly, as assumed in planning the search, no information is available when the pause is made that was not available when the original plan was constructed. Hence, if the plan is optimal, it is consistent, under these conditions.

Optimality also implies consistency when we admit other contingencies, provided that they can be foreseen and included in the search plan. For example, if the region contains one false target in addition to the one target, the optimal search plan may depend on the state of detection of the false target, which can be included in the search plan. Thus, the optimal search plan is consistent, but not contingent-free, in this case.

The optimal search plan might not be consistent when a pause of positive duration is made. For example, suppose that a golfer, while looking for a lost ball, tramples the ground and hits weeds with his club and then sits down to rest and plan his further search. If he doesn't sit too long, the position of the ball will not be changed, and the grounds and weeds will be in essentially the same state when he resumes searching

as they were when he sat down to rest. However, the conditions of the grounds and weeds could be altered considerably in an interval of days, by partial recovery or by changes of the type produced by a thunderstorm. Also, in a long interval the position of the ball could be changed or the ball might be removed from the region entirely.

We avoid these difficulties by assuming that the target is stationary and the conditional detection function has an additive property. Let $m(x)$ be the search density at a point x of the region R, assumed to be Euclidean n-space. Assume that the conditional detection function, called the local effectiveness function by Stone and Stanshine [4], is a function $b(m(x))$ of the search density function alone. (We could assume that b is a function of x, as well, but the gain in generality is slight.) Thus, $b(m(x))$ depends only on the total density $m(x)$ and is not affected by the passage of time during which no searching is done. We assume that $b(m)$ has the usual properties of a probability function. For convenience, we also assume that it has a derivative that is positive, continuous, and strictly decreasing.

Assume that we search with density $m_1(x)$, do not find the target, and then search with the additional density $m_2(x)$. Let

$\hat{b}(m_2(x)|m_1(x))$ = conditional probability of detecting a target at x with a search of density $m_2(x)$, given that it was not detected by a previous search of density $m_1(x)$

By conditional probabilities we have

$$b(m_1(x) + m_2(x)) = b(m_1(x)) + [1-b(m_1(x))]\,\hat{b}(m_2(x)|m_1(x))$$

from which we obtain the equation

$$\hat{b}(m_2(x)|m_1(x)) = \frac{b(m_1(x)+m_2(x))-b(m_1(x))}{1-b(m_1(x))} \qquad (1)$$

Let $f(x)$ be the density function for the target in the region R. Then the probability of detection with the search density function $m_1(x)$ followed by $m_2(x)$ is

$$P(m_1+m_2) = \int_R f(x)b(m_1(x)+m_2(x))dx \qquad (2)$$

The global conditional detection function $\hat{P}(m_2|m_1)$, given that the target was not detected by the search of density $m_1(x)$, is obtained by an equation analogous to equation (1) for the local conditional detection function.

Equation (2) shows that $P(m)$ is an additive functional of m. Put

$$\beta(m(x)) = 1-b(m(x)), \ \hat{\beta}(m_2(x)|m_1(x)) = 1-\hat{b}(m_2(x)|m_1(x))$$

Then equation (1) takes the simpler form

$$\hat{\beta}(m_2(x)|m_1(x)) = \beta(m_1(x) + m_2(x))/\beta(m_1(x)) \qquad (1')$$

We say that $\hat{\beta}$ is additively dependent on the previous searching effort when equation (1') holds.

In some search problems a contingency may occur that prevents the use of equations (1) and (2) when $m_1(x)$ is applied before the contingency occurs and $m_2(x)$ afterward. For example, suppose that we are searching for an object that was lost in the sand on a beach that is exposed, and available to be searched, only at low tide. The effects produced by digging and sifting are undisturbed while the tide is out, but are undone by the next high tide. The conditional detection function is additive for the search that is made during any one low tide, but not for two low tides. If we assume that searches of densities $m_1(x)$ and $m_2(x)$, made in two low tides are completely independent, equation (2) is replaced by

$$P(m_1, m_2) = 1-[1-P(m_1)][1-P(m_2)] \qquad (3)$$

where the detection probability P now is a functional of the two density functions, m_1 and m_2.

A different equation is obtained if we assume that the target location is not changed by the action of the high tides, that we can locate the density $m_2(x)$ relative to $m_1(x)$, and that the local conditional detection function in the second low tide is the same as that in the first low tide. Then equations (1) and (2) are replaced by

$$\hat{b}(m_2(x)|m_1(x)) = b(m_2(x)) \tag{4}$$

$$P(m_1, m_2) = 1 - \int_R f(x)[1-b(m_1(x))][1-b(m_2(x))]dx \tag{5}$$

Here, $m_1(x)$ is the density of effort during the first low tide and $m_2(x)$ is the density of effort during the second low tide; no other division of $m_1(x) + m_2(x)$ can be used in equations (4) and (5). When the conditional detection function satisfies equation (4) we say that it is independent of the effort in the previous stages.

Another search problem of this type is that of an integrating detector that recovers rapidly between successive "looks" at targets. While a target is in the beam of the detector the signals received by the detector are additive; the response of the detector to one look at a target depends only on the sum of the signals received in that look. However, the conditional probability of detecting a target at x in a second scan of the target space is obtained from equation (4), assuming complete recovery of the detector. The probability of detection in two scans of the sensor, with densities $m_1(x)$ and $m_2(x)$, is obtained from equation (5).

Equations (4) and (5) were assumed by J. M. Danskin [1]. The conditional detection function b(m) is additive in each separate stage (low tide in the beach problem and scan for the integrating detector), but is not additive for two or more stages. The problem of maximizing $P(m_1, m_2)$ in equation (5) for a given total effort is not essentially different from the problem of maximizing $\hat{P}(m)$ when it satisfies equation (2). If the effort is specified for each stage, the problem can be solved by Dynamic Programming or by a method devised by Danskin.

Although the optimal search plan is not contingent-free, the contingency for the beach problem and for the integrating detector can be foreseen and included in the search plan. Again, the optimal search plan is consistent, provided it is followed exactly.

If the optimal search plan is not followed exactly, we may be able to improve the search plan by replanning at intervals, provided we can determine and record the actual search density that has been applied. It is well known that the actual effort distribution does not coincide with the planned distribution, and sometimes the difference is large. Also, it often is difficult to determine what distribution has been applied, and to make provisions for recording this information and using it in replanning. We will return to this question in formulating the false target problem.

If $b(m)$ is the negative exponential function, $\hat{b}$ in equation (1) also is the negative exponential function, and equation (4) is satisfied. The negative exponential function is the only probability function with this property. For this conditional detection function the probability functionals (2) and (5) are identical. DeGuenin [2] replaced the negative exponential function by more general functions that are additive and hence satisfy equation (1), while Danskin [1] generalized by assuming equation (4).

2. The False Contact Search Problem

We start with a general discussion of false contacts and problems that are produced by the false contacts, before restricting the discussion to false contacts that are generated by real stationary objects.

False contacts may be generated by real objects that can't be distinguished from the target with certainty, except by a close inspection. For example, in search by active sonar for a submerged submarine some possible generators of false contacts are wrecks on the bottom, sea mounts and bottom irregularities that reflect sound waves, whales, other marine life, and decoys (including bubble clouds produced by the interaction with water of chemicals expelled from the target submarine).

False contacts also may be generated by anomalies or variations (usually called noise) in the signal response, produced by fluctuations in the performance of the detection equipment and fluctuations in the medium through which the energy passes.

When a contact is made the searcher may decide to take no immediate action, while continuing to observe the contact indication; or, decide to stop searching and investigate the contact; or, decide to proceed with appropriate action, on the assumption that the contact is the target. If no immediate action is taken, the searcher may decide later to investigate the contact or to take some other action. Under combat conditions it may not be feasible to investigate the contact; the searcher can ignore the contact, attempt to maintain contact, or make an attack.

As the search proceeds the searcher may obtain additional information on the generators of false contacts, their types, numbers, and locations. Some of the information will be obtained by direct observation and some of it by deduction. For example, the number and locations of contacts, and the times at which the contacts were made, can be observed and recorded. Also, results of investigations, if made, would be known. Inferences can be made about the number and location of false targets not contacted, under some conditions, from the observed results.

The searcher may be able to use some of the additional information in the continuation of the search. In general, the extent to which he can do so will depend on the contingencies that have been anticipated and provided for in the search plan, and the availability of the required information. For example, the searcher may be using a search plan that anticipates the finding of real objects that generate false contacts. Each time such an event occurs the search plan may change in a way that requires the estimation of the posterior distribution of residual false targets. The posterior distribution of residual false targets depends on the distribution of effort that the searcher has applied up to that epoch. The applied distribution of effort almost always differs from the planned distribution, and often by large amounts. Hence, to estimate the posterior

distribution of residual false targets under realistic assumptions it is necessary to record the actual distribution of search effort. If no provision is made for the collection of this information and for its use in computing the posterior distribution, a different search plan must be used. (In this connection we observe that a search plan that requires the effort to be distributed in time, as well as in location, would require continuous feedback from the actual search distribution to the search plan, and would be very difficult to implement.)

The above discussion of the false contact search problem has been made to provide a proper perspective, by showing the complexity of the general problem before turning to a special case. False contacts may be generated by real objects and by random fluctuations. Real objects may be stationary or moving, and may or may not be identifiable. Various action options are available to the searcher when contacts occur. Additional information that becomes available as the search proceeds may be used to improve the search plan, provided the search plan includes the relevant contingencies. It is evident that a mathematical abstraction that includes all the possibilities would be very complicated. We start with a special case that illustrates some of the difficulties.

3. False Contacts Generated by Real Stationary Objects

Stone and Stanshine [4] assume that false contacts are generated by real objects that can be marked, when located and identified, so that they will not require investigation should they be contacted again. They also assume that the false contacts are generated as an inhomogeneous Poisson process. We generalize the Stone-Stanshine model in several respects.

We assume that exactly one stationary target is known to be in a region R of Euclidean n-space, with known location density function $f(x)$. During the search, false contacts may be generated by other real objects in R, called false targets, which can't be distinguished from the target except by a close inspection. The number of false targets in R may be known, or unknown with a known number density function. We assume that the false

targets are independently distributed with known location density functions. They may be identically distributed with common location density function $g(x)$, or otherwise. Our assumptions are more general than those of Stone and Stanshine, who explicitly introduce only the collective location density function $\delta(x)$. Other properties of the false target population must be inferred from the assumptions on false contacts, by means of which one can deduce that their number density function for false targets is the Poisson function.

If a contact is made, the search will be interrupted, an investigation will be started, and continued until the contact has been identified. If the contact is the target, the search will be stopped. If the contact is a false target, its location will be recorded and the position marked, perhaps with a buoy, so that another investigation will not be made, should it be contacted again. Then search will be resumed.

Under these conditions the optimal search plan may depend on the number of false targets that have been found and eliminated, since the distribution of the residual (undetected) false targets may be changed as false targets are found and eliminated. Hence, in formulating the search problem and writing the expression for the optimization functional we should include the number of found false targets as a contingency, unless we can demonstrate in advance that the optimal search plan is independent of the number of found false targets.

Stone and Stanshine [4] formulate their search problem and write an equation for the expected time to find the target under the implicit assumption that the search plan does not depend on the number of found false targets. They solve this problem and then undertake to prove that their solution has the assumed property. We question the validity of this argument. The expected-time functional when the search plan is assumed to be contingent on the number of found false targets is not the same as (or even remotely similar to) the corresponding functional when the search plan is assumed to be independent of the contingency. Hence we have no assurance that we are optimizing the correct functional unless we can show

in advance that the optimal search plan is independent of the number of found false targets. For the special conditions assumed by Stone and Stanshine a prior proof can be made, as shown in section 5 below.

The posterior proof given by Stone and Stanshine is incomplete. In section 4, they use "additivity" to describe the properties that we call "consistency" and "contingent-free". They state that the optimal plan is to continue with the original search plan, if at any time before the target has been found it is decided to replan the search to minimize the expected time remaining to find the target. To show this they examine two contingencies, as follows:

(a) No contact has been made,

(b) A contact has been made and not investigated.

But these contingencies are included in the formulation of the problem and it was proved in section 3 of their paper that immediate contact investigation is optimal. If it were possible to improve the search plan under contingencies (a) and (b), the search plan would not be optimal. On the other hand, they fail to consider the contingency

(c) A contact has been made, investigated, and found to be false.

Since this contingency was not included in the original formulation, the search plan that is optimal when it is not included conceivably could be improved by including it. Only for the overlooked contingency (c) does optimality not imply consistency.

4. Effort Distributions and Conditional Detection Functions

It will be convenient to use a notation that permits the search density function to change each time a false target is found. Let

s_1 = search time at which the first contact is made,

$\mu_1(x,s)$ = search density at x by search time s, $0 \le s \le s_1$.

If the contact is a false target and search is resumed, let

s_2 = additional search time in the second stage, measured from the resumption of search until a second contact is made,

$\mu_2(x,s)$ = additional search density at x by the additional search time s in the second stage, $0 \leq s \leq s_2$.

In general, let

s_i = additional search time to make a contact in the i^{th} stage of the search,

$\mu_i(x,s)$ = additional search density at x by the additional search time s in the i^{th} stage, $0 \leq s \leq s_i$.

We assume that $\mu_i(x,0) = 0$ and that $\mu_i(x,s)$ is nondecreasing in s for each i and for every x in R. The total effort density at x in the first j stages is

$$M_j(x) = \sum_{i=1}^{j} \mu_i(x,s_i) \tag{6}$$

At additional search time s in the j^{th} stage, after (j - 1) false targets have been found, the total search density at x is

$$m_j(x,s) = M_{j-1}(x) + \mu_j(x,s) \tag{7}$$

We call attention to the fact that $M_j(x) \equiv m_j(x,s_j)$ in our notation. Thus, capital M is used for the value of m when contact occurs. The variable s is used for the additional search time; the total search time S corresponding to the additional search time s in the j^{th} stage is $S = s_1 + s_2 + \ldots + s_{j-1} + s$.

The total effort that has been applied in the additional search time s in the j^{th} stage is

$$w_j(s) = \int_R \mu_j(x,s)\, dx$$

and the total effort $W_j(S)$ from the start of search is

$$W_j(S) = w_j(s) + \sum_{i=1}^{j-1} w_i(s_i) \ , \ S = s + \sum_{i=1}^{j-1} s_i$$

Then we can impose the condition that the effort in the j^{th} stage, or the total effort, is a known function of the search time. For example, the condition that effort is applied at a constant rate U_j in the j^{th} stage is

$$w_j(s) = s\, U_j \tag{8}$$

We assume that the conditional probability that a target at x will be contacted when an effort of total density m(x) is applied at x is a function b(m(x)), called the local effectiveness function by Stone and Stanshine, of the total density m(x), however it is applied. We assume that $b(0) = 0$, $\lim b(u) = 1$ as $u \to \infty$, and that the derivative $b'(u)$ is positive, continuous, and strictly decreasing. Some of these properties can be relaxed or omitted under some conditions.

We obtain the conditional contact function at x for the j^{th} stage, given that the effort of density $M_{j-1}(x)$ was applied in the first $j - 1$ stages, by using our assumption that b is a function of the total density. Let

$\hat{b}_j(x,s) \equiv b(\mu_j(x,s)|M_{j-1}(x))$ = conditional probability that a target at x will be contacted by the additional effort of density $\mu_j(x,s)$ in the j^{th} stage, given that it hasn't been contacted by the effort of density $M_{j-1}(x)$ that has been applied in the previous stages.

Put

$$\beta(m(x,s)) = 1 - b(m(x,s)), \ \hat{\beta}_j(x,s) = 1 - \hat{b}_j(x,s)$$

By the argument used to obtain equations (1) and (1') we have

$$\hat{\beta}_j(x,s) = \frac{\beta(\mu_j(x,s)+M_{j-1}(x))}{\beta(M_{j-1}(x))} \tag{9}$$

Let $\alpha(u)$ be the conditional contact failure function for false targets, corresponding to $\beta(u)$ for the target. Then the conditional contact failure probability $\hat{\alpha}_j(x,s)$ in the j^{th} stage is obtained from the equation

$$\hat{\alpha}_j(x,s) = \frac{\alpha(\mu_j(x,s)+M_{j-1}(x))}{\alpha(M_{j-1}(x))} \tag{10}$$

5. Posterior False Target Distributions

We assume, at the start, that the search effort distribution is known exactly. Also, we assume that the false targets are independently distributed in R, and we restrict attention to the case in which they are identically distributed with known location density function g(x) and known number density function p(n). We want to find the posterior location density function $\hat{g}_j(x,s)$ and the posterior number density function $\hat{p}_j(n,s)$ at the time s after the start of search in the j^{th} stage, given that j-1 false targets have been found and eliminated.

Under the assumed conditions we can obtain the posterior location and number density functions by the direct application of conditional probabilities, sometimes referred to as Bayes' theorem. We could do this for the j^{th} stage directly. However, we prefer to proceed step by step from the start of search in the first stage to the start of search in the second stage, to see how the distributions change as information becomes available. By repeating the argument we can obtain the distributions for any desired stage.

In the first stage the probability that a false target chosen at random is not contacted in search time s is

$$Q_1(s) = \int_R g(x)\alpha(\mu_1(x,s))\,dx \tag{11}$$

for $0 < s \leq s_1$. The posterior location density function for false targets is

$$\hat{g}_1(x,s) = g(x)\alpha(\mu_1(x,s))/Q_1(s) \tag{12}$$

and the posterior number density function is

$$\hat{p}_1(n,s) = p(n)Q_1^n(s) / \sum_{i=0}^{\infty} p(i)Q_1^i(s) \text{ , } n = 0, 1, 2, \ldots \qquad (13)$$

by the direct application of conditional probabilities.

Assume that a contact is made at location x_1 and search epoch s_1, investigated, and found to be false. This event has no effect on the location distribution of the unlocated false targets, since they are assumed to be independently and identically distributed; their location density function is $\hat{g}_1(x,s_1)$ at the start of search in the second stage. However, the event may have an effect on the number distribution. Define the event A to be

Event A: a false target is at x_1 and is contacted at search epoch s_1; no other contacts are made in search time s_1.

Let

$P\{A|n+1\}$ = probability of event A, given that there were n+1 false targets initially in R

To obtain the probability that a false target is "at x_1" and is contacted "at search epoch s_1", we need to define the terms in quotation marks. We say that a false target is at x_1, if it is in an arbitrarily small interval dx centered at x_1; and that it is contacted at epoch s_1, if it is not contacted in search time s_1 and is contacted in the arbitrarily small interval $(s_1, s_1 + ds)$. Then, to first order terms in dx and ds,

$$P\{A|n+1\} = \Big[(n+1)g(x_1)dx\Big] \Big[Q_1^n(s_1)\alpha(\mu_1(x_1,s_1))\Big] \Big[\alpha(\mu_1(x_1,s_1))-\alpha(\mu_1(x_1,s_1+ds))\Big] \qquad (14)$$

The first quantity in square brackets is the probability that a false target is at x_1, the second is the probability that none of the n+1 false targets is contacted in the search time s_1, and the third is the probability that the false target at x_1 is contacted in the interval $(s_1, s_1 + ds)$.

The probability that there are n residual false targets in R at the start of search in the second stage, given that one has been contacted at x_1 at epoch s_1 and eliminated, is

$$\hat{p}_2(n,0) = p(n+1)\ P\{A|n+1\} \ / \ \sum_{i=0}^{\infty} p(i+1)\ P\{A|i+1\} \ ,$$

from which we obtain

$$\hat{p}_2(n,0) = (n+1)p(n+1)\ Q_1^n\ (s_1) / \sum_{i=0}^{\infty} (i+1)p(i+1)Q_1^i\ (s_1),\ n = 0,1,2,\ldots \quad (15)$$

We have divided numerator and denominator by common factors.

We now have completed one cycle, and can repeat the above argument to get $\hat{g}_2(x,s)$ and $\hat{p}_2(n,s)$. The location density function of unlocated false targets at the start of search in the second stage is

$$\hat{g}_2(x,0) = \hat{g}_1(x,s_1)\ , \quad (16)$$

which can be obtained by putting $s = s_1$ in equation (12). Using conditional probability arguments we obtain

$$\hat{g}_2(x,s) = \hat{g}_2(x,0)\ \hat{\alpha}_2(x,s) / \hat{Q}_2(s)\ , \quad (17)$$

where $\hat{\alpha}_2(x,s)$ is obtained from equation (10) and

$$\hat{Q}_2(s) = \int_R \hat{g}_2(x,0)\ \hat{\alpha}_2(x,s)dx = Q_2(s)/Q_1(s_1)\ , \quad (18)$$

$$Q_2(s) = \int_R g(x)\alpha(\mu_1(x,s_1) + \mu_2(x,s))\ dx \quad (19)$$

The second equality in (18) is obtained from equations (10), (12), and (16). Substituting from equations (16), (10), and (18) we obtain from equation (17) the posterior location density function

$$\hat{g}_2(x,s) = g(x)\alpha(\mu_1(x,s_1) + \mu_2(x,s))/Q_2(s) \quad (20)$$

Similarly, the posterior number density function at additional search time s in the second stage is

$$\hat{p}_2(n,s) = \hat{p}_2(n,0)\ \left[\hat{Q}_2(s)\right]^n \ / \ \sum_{i=0}^{\infty} \hat{p}_2(i,0)\ \left[\hat{Q}_2(s)\right]^i$$

Using equations (15) and (18) we obtain

$$\hat{p}_2(n,s) = (n+1)p(n+1)\ Q_2^n(s)/\sum_{i=0}^{\infty}(i+1)p(i+1)Q_2^i(s),\ n = 0,1,2,... \quad (21)$$

By induction we obtain the location and number density functions for the j^{th} stage, given that j-1 false targets have been found and eliminated. The location density is

$$\hat{g}_j(x,s) = g(x)\alpha(m_j(x,s))/Q_j(s) \quad (22)$$

and the number probability density is

$$\hat{p}_j(n,s) = (n+1)(n+2)...(n+j-1)p(n+j-1)Q_j^n(s)/\Sigma_j(s),\ n=0,1,2,... \quad (23)$$

where

$$Q_j(s) = \int_R g(x)\alpha(m_j(x,s))\ dx \quad (24)$$

and

$$\Sigma_j(s) = \sum_{i=0}^{\infty}(i+1)(i+2)...(i+j-1)p(i+j-1)Q_j^i(s) \quad (25)$$

The location density function $\hat{g}_j(x,s)$ depends on j in a superficial way, since j is involved only in the total effort density function $m_j(x,s)$ that has been applied. The same location density function would be obtained for a given total effort density function, regardless of the number of stages and the distribution of effort among stages. However, $\hat{p}_j(n,s)$ in equation (23) depends on j in a fundamental way, in general. An exception is that in which p(n) is a Poisson function,

$$p(n) = e^{-N}\ N^n/n\ !$$

having expected value N, say. Then $p_j(n,s)$ in equation (23) reduces to

$$\hat{p}_j(n,s) = e^{-NQ_j(s)}\left[NQ_j(s)\right]^n/n\ !\ ,$$

the Poisson density function with expected value $NQ_j(s)$. Here, again, j is involved in a superficial way.

If $\hat{p}_j(n,s)$ depends on j only with respect to the total effort density function $m_j(x,s)$, it is not necessary to change the search plan each time a false target is found and eliminated. The Poisson density function has this property. It is the function assumed (implicitly) by Stone and Stanshine [4]. Are there other density functions that have this property?

Equations (22) and (23) apply when the false targets are independently and identically distributed, with known location density function g(x), and known number probability density function p(n); and we search with known search density function $m_j(x,s)$, and known local effectiveness function $a(m) = 1 - \alpha(m)$. Important search problems occur for which at least one of these assumptions is not valid.

For example, suppose that the false targets are wrecks that are accurately located in earth coordinates, while the searcher's position is not accurately known in earth coordinates. Then the false targets are not independently (or identically) distributed relative to the searcher. The location of a found false target yields information on the location of the other false targets, and this information may be sufficient to locate the other false targets with an accuracy that would permit them to be avoided in further search.

The region R may contain several classes of false targets, such as wrecks and bottom irregularities. If the investigation is sufficient to determine the class to which the false target belongs, as well as the fact that it is not the target, we can treat each class separately. If the investigation only establishes the fact that the contact is not the target, the formulation is more complicated; we need to estimate the probabilities that the false target belongs to the various classes from the collective class densities at the location and time of contact.

Equations (22) and (23) for the location density function and the number probability density function require that we know the search density $m_j(x,s)$ that has been applied. In many search problems it is difficult

to determine $m_j(x,s)$ accurately, even when provision is made for the recording of the tracks of the searchers. If $m_j(x,s)$ is not known, we can't apply the Bayesian arguments used to get equations (22) and (23). Even when provision is made to record the tracks of the searchers, there may be a large uncertainty about $m_j(x,s)$, which then should be treated as a stochastic, rather than a fixed, function. Usually we assume that $m_j(x,s)$ is equal to the planned density $m_j^*(x,s)$, without any means of measuring and recording the actual distribution.

If the uncertainty in $m_j(x,s)$ can be expressed in the form of a density function for one or several parameters, an argument similar to the one used to derive equations (22) and (23) perhaps could be made to obtain replacements for these equations. If no record of $m_j(x,s)$ is kept, there appears to be no reasonable alternative to the use of equations (22) and (23) with the planned density $m_j^*(x,s)$.

6. Optimization Criteria

It usually is assumed that the general objective of the search is to find the target quickly. Two precise criteria that are obtained from this general objective are:

(1) To maximize the probability P(t) of finding the target by a given elapsed time t;

(2) To minimize the expected time to find the target.

An objection that can be raised to the second criterion is that it assumes implicitly that search will be continued until the target is found, a condition that the searcher will not always adhere to in practice, even when he accepts it as a condition in planning the search. Also, there is a possibility that our assumption that there is a target in R is not valid.

When no false targets are present we can reconcile the two criteria. If a search plan exists that maximizes P(t) for all t, it minimizes the expected time to find the target. Such a plan exists when no false targets are present. Of course, it is much more difficult to

lay out a practical search plan that approximates the distribution of the ideal solution when trying to maximize $P(t)$ for all t than when trying to maximize $P(t)$ for a single value of t.

When false targets are present we have no assurance that a plan exists that maximizes $P(t)$ for all t. In section 5 of their paper [4] Stone and Stanshine exhibit an example for which the search plan that maximizes $P(t)$ for a particular value of t does not coincide with the search plan that minimizes their integral $\mu(m)$ for the expected time, from which they conclude that no plan maximizes $P(t)$ for all t, since such a plan must coincide with the plan that minimizes the expected time. However, they do not write an equation for $P(t)$ or for the expected-time functional, say $\bar{t}(m)$, that is obtained from $P(t)$. To complete the argument it is necessary to prove that $\bar{t}(m) = \mu(m)$ for all m, since these two functionals are derived by quite different arguments and do not appear to be identical. A proof that is simple, but not obvious, can be made.

The optimization functional is difficult to write for both criteria when false targets are present. In the next section we show how to construct the expected-time functional when the number of false targets is a known finite number n, and we outline an optimization procedure. An obvious modification of the procedure can be used when the number is not known but has a density function $p(n)$ that is restricted to finite values, that is, there exists a K such that $p(n) = 0$, $n > K$. The problem has not been solved for non-finite distributions.

We also have examined the formulation for criterion (1), that of maximizing the probability of detection by a given time. The problem appears to be as difficult as, and perhaps more difficult than, the problem for criterion (2), which is a better reason for starting with the expected-time criterion than the fact that no plan maximizes $P(t)$ for all t.

7. Expected-Time Criterion

We start with the expected-time criterion (2). Our search procedure is as follows:

a. Search until a contact occurs, investigate the contact until it is identified.

b. If the contact is the target, stop the search.

c. If the contact is a false target, record the location and mark the site so that investigation will not be repeated, should this false target be contacted again. Increase the number of found false targets by 1 and use the fact that a false target has been found and eliminated to adjust the number probability density of residual false targets and the location distributions of residual false targets, to the extent possible. Then continue the search.

d. Repeat the above steps until the target is found.

If the initial number n of false targets is known, the number of residual false targets in the j^{th} stage is $n - j + 1$. Let

s_j = additional search time in the j^{th} stage at which contact is made in that stage;

I_j = time to investigate the j^{th} contact, if false;

q_j = conditional probability that the j^{th} contact is false, given that a contact has occurred at x_j in the j^{th} stage.

Then the time t_c to contact the target is

$$t_c = s_1 + q_1(I_1 + s_2 + q_2(I_2 + s_3 + q_3(\ldots + q_n(I_n + s_{n+1}))\ldots) \tag{26}$$

Let

$$\left.\begin{aligned} T_{n+1} &= s_{n+1} \\ T_n &= s_n + q_n(I_n + T_{n+1}) \\ T_{n-1} &= s_{n-1} + q_{n-1}(I_{n-1} + T_n) \\ &\vdots \\ T_1 &= s_1 + q_1(I_1 + T_2) \end{aligned}\right\} \tag{27}$$

Then $t_c = T_1$ and the elapsed time t to find the target is $t = t_c + I$, where I is the time to investigate the contact that is the target.

The expected time to find the target is $\bar{t} = \bar{t}_c + \bar{I}$. Since $\bar{I}$ is independent of the search plan, we minimize $\bar{t}_c$. The expected time $\bar{t}_c$ to contact the target is obtained by averaging stage by stage in equations (27), starting with the $(n+1)^{st}$ stage and moving (backwards) to the first stage.

In the $(n+1)^{st}$ stage there are no false targets. The density function at the start of search in the $(n+1)^{st}$ stage is

$$f_{n+1}(x) = \frac{f(x)\beta(M_n(x))}{Q_n}, \tag{28}$$

where $M_n(x)$ is obtained from equation (6) and

$$Q_n = \int_R f(x)\beta(M_n(x))\, dx \tag{29}$$

We now find the function $\mu^*_{n+1}(x,s)$ that minimizes the expected value of s_{n+1}, which is a solved problem. The optimal function $\mu^*_{n+1}(x,s)$ is the function $\mu_{n+1}(x,s)$ that minimizes the failure probability $Q_{n+1}(s)$ for all s, where

$$Q_{n+1}(s) = Q_n^{-1} \int_R f(x)\beta(\mu_{n+1}(x,s) + M_n(x))\, dx \tag{30}$$

Let $Q^*_{n+1}(s)$ be the function $Q_{n+1}(s)$ when μ_{n+1} has been replaced by the optimal function μ^*_{n+1}, and put

$$T^*_{n+1} = \int_0^\infty Q^*_{n+1}(s)\, ds \tag{31}$$

Then T^*_{n+1} is the minimum expected time to contact the target in the $(n+1)^{st}$ stage. From equations (6) and (30) it is seen that T^*_{n+1} may depend on the previous density functions $\mu_j(x,s)$ and the times s_j at which previous contacts were made, $j = 1, 2, \ldots, n$.

In the n^{th} stage we minimize the expected time remaining to contact the target, on the assumption that the search procedure in the $(n+1)^{st}$ stage will be optimal. Hence, we minimize the expected value of

$$s_n + q_n(I_n + T^*_{n+1}) \equiv T_n(s_n, \mu_n) \tag{32}$$

That is, we find the function $\mu^*_n(x,s)$ that minimizes $\overline{T}_n(s_n, \mu_n)$ over s_n, given that there is one false target and one real target in R. The expected value is computed over the probability distribution of the false contact time s_n, noting that q_n and T^*_{n+1} (and possibly I_n) depend on s_n and μ_n, in general. This is not a solved problem. We have solved it in special cases used as examples, one of which will be shown later.

Assume that the above problem has been solved and let T^*_n be the minimum value of the expected value of $T_n(s_n, \mu_n)$. Then in the $(n-1)^{st}$ stage we minimize the expected value of

$$s_{n-1} + q_{n-1}(I_{n-1} + T^*_n) \equiv T_{n-1}(s_{n-1}, \mu_{n-1}) \tag{33}$$

This problem is similar to the optimization problem in the previous stage, except that there now are two false targets. Also, the functional form of T^*_n is more complicated than that of T^*_{n+1}.

Continuing in the way outlined above we find the functions $\mu^*_{n+1}(x,s)$, $\mu^*_n(x,s)$ ... , $\mu^*_1(x,s)$, which constitute the optimal solution, in the sense of minimizing the expected time to find the target when all the available information is used. The function $\mu^*_j(x,s)$ in the j^{th} stage may depend on the functions μ^*_1, μ^*_2, ... , μ^*_{j-1} and the contact times s_1, s_2, ... , s_{j-1} in the previous stages. These functions and times will be known when search is resumed in the j^{th} stage. The optimal function $\mu^*_j(x,s)$ in the j^{th} stage is found under the assumption that search in the later stages will be optimal.

If the initial number n of false targets is not known but has a finite distribution, we can apply a similar procedure to find the solution. If the maximum value of n is K, there are no false targets in the $(K+1)^{st}$ stage. In the K^{th} stage the possible number n_K of residual false targets is 0 or 1. If $n_K = 0$, $q_K = 0$ and there are no later stages. If $n_K = 1$,

the problem is the same as the problem when n is known. We find the expected value of $T_K(s_K, \mu_K)$ over the distribution of n_K before proceeding with the optimization; similarly for the other stages. Thus, in principle, we can find the optimal solution in (K+1) steps by applying the general principle of Dynamic Programming.

If the distribution of n is not finite, the problem is more difficult. There now is no convenient starting place. We do not know how to solve this problem.

8. Solution For a Particular Problem

We find the solution for a simple example to show the general procedure that is used in minimizing $\overline{T}_n(\mu_n, s_n)$. Assume that R consists of two disjoint regions I and II in the plane, each having Lebesque measure 1; the target density function is $f(x) = f_1$ for x in I, $f(x) = 1 - f_1$ for x in II; and there is one false target, known to be in region II. Assume that the time required to identify a contact is 1, that the conditional detection function is $b(z) = 1 - \exp(-z)$ for the target and false target, and that effort is applied at a constant rate $U = 1$.

We search with density function $\mu_1(x,s)$ until a contact is made at $s = s_1$. If the contact is the false target, we go to the second stage of the search and search with density function $\mu_2(x,s)$, knowing there are no false targets remaining. At the start of the second stage the conditional target density function has the form

$$\hat{f}_2(x) = f_2 \text{ for x in I, } \hat{f}_2(x) = 1 - f_2 \text{ for x in II}$$

where f_2 is independent of x, but depends on μ_1 and s_1. (We can obtain f_2 from equation (28) when needed.)

The search density function $\mu_2^*(x,s)$ that minimizes the expected additional time s_2 to contact the target depends on the magnitude of f_2. If $f_2 \leq 0.5$, μ_2^* has the form:

$$\text{For } 0 < s \le s_0,\ \mu_2^*(x,s) = \begin{cases} 0 \text{ for } x \text{ in I} \\ s \text{ for } x \text{ in II} \end{cases}$$

$$\text{For } s > s_0,\quad \mu_2^*(x,s) = \begin{cases} (s - s_0)/2 \text{ for } x \text{ in I} \\ (s + s_0)/2 \text{ for } x \text{ in II} \end{cases}$$

where

$$s_0 = \ln(f_2^{-1}-1)$$

From equation (30) $Q_2(s)$ for the optimal distribution is

$$Q_2^*(s) = \begin{cases} f_2 + (1-f_2)e^{-s} & , s \le s_0 \\ 2f_2^{1/2}(1-f_2)^{1/2}e^{-s/2} & , s > s_0 \end{cases}$$

Then

$$T_2^* = \int_0^\infty Q_2^*(s)ds = 1 + f_2(2 + s_0)$$

(If needed, we can get T_2^* when $f_2 > 0.5$ by replacing f_2 by $1 - f_2$ in the above expression.)

Write $\mu_1(x,s)$ in the form

$$\mu_1(x,s) = \begin{cases} y(s) & \text{for } x \text{ in I} \\ s-y(s) & \text{for } x \text{ in II} \end{cases}$$

From equation (28), f_2 becomes

$$f_2 = f_1e^{-y(s_1)}/Q_1(s_1,\ y(s_1))$$

where s_1 is the observed time at which the first contact was made, and

$$Q_1(s,y) = f_1e^{-y} + (1 - f_1)e^{-s+y} \tag{34}$$

Then

$$s_0 = 2y(s_1) - s_1 + \ln (f_1^{-1}-1)$$

and we now can write T_2^* in terms of s_1 and $y(s_1)$.

To write the equation for T_1 in equation (32') we need the conditional probability q_1 that the first contact is made on the false target, given that the contact is made at search time s_1. The probability that the false target, known to be in region II, will not be contacted by search time s is

$$Q_F(s,y) = e^{-s+y}, \tag{35}$$

where it is understood that $y = y(s)$ is a function of s. The probability that a contact will occur by search time s is

$$P_c(s,y) = 1 - Q_1(s,y)\, Q_F(s,y)$$

We note that $P_c(s,y)$ is the probability distribution function of s_1, the time of first contact. The rate of making contact at search time s is

$$P_c' = \frac{d}{ds} P_c(s,y(s)).$$

We can write P_c' in the form

$$P_c' = Q_1\, P_F' + Q_F\, P_1' \tag{36}$$

where

$$P_1 = 1 - Q_1,\ P_F = 1 - Q_F$$

and the prime indicates total derivative with respect to s. The two terms in the righthand member of equation (36) are the contact rates on the false target and the target respectively. If the first contact occurs at search time s_1, the conditional probability that the contact is made on the false target is $q_1(s_1,y(s_1))$, where

$$q_1(s,y(s)) = Q_1(s,y(s))\, P_F'(s,y(s))/P_c'(s,y(s))$$

The time $T_1(s_1, \mu_1)$ in equation (32) will be written in the form $T_1(s_1, y(s_1))$. Then

$$T_1(s,y) = s + P'_F(s,y)\ R\ (s,y)/P'_c(s,y),$$

where

$$R(s,y) = f_1 e^{-y}(2y - s + 4 + \ln(f_1^{-1}-1)) + 2(1 - f_1)e^{-s+y} \tag{37}$$

The mean value of $T_1(s_1,y(s_1))$ over s_1 is

$$\overline{T}_1(y) = \int_0^\infty T_1(s,y(s))P'_c(s,y(s))ds = \int_0^\infty F(s,y(s),y'(s))ds \tag{38}$$

where

$$F(s,y,y') = G(s,y) + y'\ H(s,y) \tag{39}$$

and

$$G(s,y) = Q_F(s,y)[Q_1(s,y) + R(s,y)]$$

$$H(s,y) = -Q_F(s,y)\ R(s,y)$$

We now find $y(s)$ to minimize $\overline{T}_1(y)$ in (38). With an integrand (39) that is linear in y' the problem is easy to solve by methods from the Calculus of Variations. The necessary conditions of Legendre and Weierstrass are satisfied. Euler's necessary condition for an extremum,

$$\frac{d}{ds}\left(\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial y},$$

reduces to the condition

$$\frac{\partial G}{\partial y} = \frac{\partial H}{\partial s}$$

with the integrand F in (39) that is linear in y'. This equation reduces to the equation

$$Q_1 + \frac{\partial Q_1}{\partial y} + \frac{\partial R}{\partial y} + \frac{\partial R}{\partial s} = 0 \tag{40}$$

with the simple form for Q_F in equation (35). When the equations for Q_1 and R in (34) and (37) are used, equation (40) becomes

$$2(f_1^{-1}-1)\ e^{2y-s} = 2y - s + 3 + \ln(f_1^{-1}-1) \ ,$$

which becomes

$$2\ e^{u} = u + 3 \tag{41}$$

with the change of function,

$$u = 2y - s + \ln(f_1^{-1}-1) \tag{42}$$

There are two solutions of equation (41). Let u_1 be the positive solution, which is approximately $u_1 = 0.583$. The corresponding solution of equation (42) is

$$y_1(s) = \frac{1}{2}\ (s + u_1 - \ln(f_1^{-1}-1)) \tag{43}$$

We now can show that the solution $y_1(s)$ in equation (43) minimizes the integral $\overline{T}_1(y)$ in equation (38) by showing that the second variation of the integral is non-negative, which is a form of Jacobi's condition. The second variation I" of the integral is

$$I'' = \int_0^\infty (F_{yy}\ \eta^2 + 2\ F_{yy'}\ \eta\eta' + F_{y'y'}\eta'^2)ds,$$

where $\eta = \eta(s)$ is an arbitrary variation from $y_1(s)$ that vanishes at the end points. With the F functional in equation (39) I" becomes

$$I'' = 4\int_0^\infty [(1-f_1)(3-2y')\ e^{2y-2s}\ \eta^2-(f_1e^{-s} + 2(1-f_1)e^{2y-2s})\eta\eta']ds$$

Integrating by parts and using the property that $\eta(s)$ vanishes at 0 and ∞, we have

$$\int_0^\infty (1-y')e^{2y-2s}\ \eta^2\ ds = \int_0^\infty e^{2y-2s}\ \eta\eta'\ ds$$

When we use this equation to eliminate y', the second variation becomes

$$I'' = 4\int_0^\infty [(1-f_1)e^{2y-2s}\ \eta^2 - f_1e^{-s}\ \eta\eta']ds$$

Again, integrating $e^{-s}\eta\eta'$ by parts, we obtain

$$I'' = 2 \int_0^\infty [2(1-f_1)e^{2y-2s} - f_1 e^{-s}] \eta^2 ds$$

For $y = y_1(s)$ in equation (43) I" reduces to

$$I'' = 2f_1(u_1 + 2) \int_0^\infty e^{-s} \eta^2 \, ds,$$

for which $I'' \geq 0$ for all $\eta(s)$, with equality only for trivial $\eta(s)$.

With the negative root u_2 of equation (41), I" is negative, since $u_2 < -2$. The corresponding function $y_2(s)$ maximizes the integral (38). Of course, it does not maximize the expected time to detection, since the minimum time T_2^* is used in finding $\overline{T}_1(y)$.

The function $y_1(s)$ in equation (43) is the unrestricted minimizing function. We obtain the optimal function $y^*(s)$ by imposing the restrictions $0 \leq y^*(s) \leq s$. The optimal solution depends on the value of f_1. If $f_1 < (1 + e^{u_1})^{-1} \doteq 0.358$,

$$y^* = \begin{cases} 0 , & 0 < s \leq - u_1 + \ln (f_1^{-1}-1) \\ \frac{1}{2}(s + u_1 - \ln(f_1^{-1}-1)), & s > - u_1 + \ln (f_1^{-1}-1) \end{cases}$$

If $f_1 > (1 + e^{u_1})^{-1}$,

$$y^* = \begin{cases} s , & 0 < s \leq u_1 - \ln (f_1^{-1}-1) \\ \frac{1}{2}(s + u_1 - \ln(f_1^{-1}-1)), & s > u_1 - \ln (f_1^{-1}-1) \end{cases}$$

We note that $f_2 = (1 + e^{u_1})^{-1}$, which is less than 0.5 since $u_1 > 0$. This outcome had been anticipated above in obtaining the solution in the second stage. Also, we note that $s_0 = u_1$.

The complete solution is the following:

First Stage

For $0 < s \leq | u_1 - \ln(f_1^{-1}-1)|$,

$$\mu_1^* = \begin{cases} 0 \ , \text{ for } x \text{ in I}, \\ s \ , \text{ for } x \text{ in II, if } f_1 < (1 + e^{u_1})^{-1} \end{cases}$$

$$\mu_1^* = \begin{cases} s \ , \text{ for } x \text{ in I}, \\ 0 \ , \text{ for } x \text{ in II, if } f_1 > (1 + e^{u_1})^{-1} \end{cases}$$

For $|u_1 - \ln(f_1^{-1}-1)| < s \leq s_1$

$$\mu_1^* = \begin{cases} \frac{1}{2}(s + u_1 - \ln(f_1^{-1}-1)) \ , \text{ for } x \text{ in I} \\ \frac{1}{2}(s - u_1 + \ln(f_1^{-1}-1)) \ , \text{ for } x \text{ in II} \end{cases}$$

If $f_1 = (1 + e^{u_1})^{-1}$, the solution is simply $\mu_1^* = \frac{1}{2}s$ for all x.

Second Stage

For $0 < s \leq u_1$, $\mu_2^* = \begin{cases} 0 \ , \text{ for } x \text{ in I} \\ s \ , \text{ for } x \text{ in II} \end{cases}$

For $s > u_1$, $\mu_2^* = \begin{cases} \frac{1}{2}(s - u_1), \text{ for } x \text{ in I} \\ \frac{1}{2}(s + u_1), \text{ for } x \text{ in II} \end{cases}$

In the second stage the total search time is $s_1 + s$ and the total effort density function is $\mu_1^*(s_1) + \mu_2^*(s)$.

We start searching in region II when f_1 is less than the critical value, 0.358, and in region I when f_1 is greater than the critical value. The critical value has been changed from 0.5 with no false targets to 0.358 by one false target in region II.

We can use the above method to solve other two-cell problems. We have performed the essential part of the analysis for the cases in which there are 0 and 2 false targets in regions I and II, and 1 false target in each region.

With this background, we have attempted to solve the two-cell problem in which there is a known number n of identically distributed false targets with the common density function: $g(x) = g_1$ for x in I, $g(x) = 1 - g_1$ for x in II. We have written the solution for the $(n+1)^{st}$ stage, the n^{th} stage, and the essential part of the $(n-1)^{st}$ stage. The general form of the solution is the same as that obtained in the example above, with some complications. Thus, in the n^{th} stage we get an equation for $y_n(s)$ of the same form as that for $y_1(s)$ in equation (43), but u_1 (now u_n) is a root of the equation

$$f_1(1 - g_1)e^u(2e^u - 3 - u) = g_1(1 - f_1)(1 - 2e^u),$$

which reduces to equation (41) when $g_1 = 0$, as in the example. It appears to be possible to write out the complete solution for this case. The solution will have the simple form of the solution in the example; that is, we search in one region exclusively or we divide the effort evenly between the two regions; no other division of the effort is admitted in the optimal solution.

If the number of false targets is not known and the distribution is known and finite, the solution becomes more complicated. We have examined the case in which $p_0 \neq 0$, $p_1 \neq 0$, $p_i = 0$, $i \geq 2$. Again, the solution has the general form described above. It appears to be possible to write out the complete solution for the two-cell problem with a finite distribution.

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13. ABSTRACT

Some search problems are described and discussed, starting with the case of no false contacts, to provide a foundation for the development of a search theory in the presence of false contacts. Important properties of the search plan are defined and illustrated by examples. After a general description of false contact generators and alternative actions that may be used when contacts are made, the problem is formulated for the case in which false contacts are generated by real stationary objects that are investigated when contacted. It is shown that the formulation and solution of the standard optimization functionals are contingent on the number of false targets found, in general, and possibly on the locations and times of contact as well. The optimization functional is difficult to write and more difficult to solve. The formulation is made for the expected-time functional when the number of false targets is limited to finite values. The solution is outlined and illustrated with an example.

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